SARAJEVO JOURNAL OF MATHEMATICS DOI: 10.5644./SJM.14.1.10
Vol.14 (27), No.1, (2018), 113–123

THE STRONG CONVEX COMPACTNESS PROPERTY AND C_0-SEMIGROUPS ON SOME HEREDITARILY INDECOMPOSABLE BANACH SPACES

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ABSTRACT. In this paper, we study the strong convex compactness property on the hereditarily indecomposable Banach spaces denoted respectively by X_{GM} and X_{AM} constructed by T. Gowers and B. Maurey (1993) and Argyros-Motakis (2014). We prove in particular that the Bochner integral of a family of strictly singular operators is a strictly singular operator. Also, some properties of C_0-semigroups defined on these Banach spaces are given.

1. INTRODUCTION

It is known that the space of compact linear operators between Banach spaces X and Y (denoted by $\mathcal{K}(X,Y)$) is not complete for the strong operator topology defined on $\mathcal{L}(X,Y)$ (the space of all bounded linear operators from X into Y). However, this norm-closed subspace has a completeness property namely the strong convex compactness property. The first author who has introduced this notion is L. Weis (1988) [26], in order to study the time asymptotic behavior of solutions for neutronic transport equations (see [20]). Notice that there exists other norm-closed subspaces of $\mathcal{L}(X,Y)$ which have this property, but not all of them [25]. Banach spaces on which the strong convex compactness property was studied have sufficiently rich structures, in other words, having unconditional basis or at least lattice structures (for example $X = L_p(\mu)(1 < p < \infty)$) or they posses norm-closed subspaces with these properties (for example $X = L_1(\mu)$ or $X = C(K)$, K compact). Since the contributions of T. Gowers and B. Maurey (1993) solving the unconditional basic sequence problem (see [14]), another category of Banach spaces was discovered, giving a dichotomy to the structure of Banach spaces (see [13]). It was shown that the space of Gowers-Maurey X_{GM} is a hereditarily indecomposable Banach space (H.I), i.e., no infinite dimensional norm-closed subspace of it

2010 *Mathematics Subject Classification.* 47A53 47A55 47D03.
Key words and phrases. Hereditarily indecomposable Banach space, strictly singular operator, spreading model, strong convex compactness property; Banach space of Gowers-Maurey X_{GM}, Banach space of Argyros-Motakis X_{AM}, C_0-semigroup.

can be decomposed into a direct sum of two further infinite dimensional norm-closed subspaces. Moreover, every bounded linear operator on X_{GM} can be written under the form $\lambda I+S$, where $\lambda\in\mathbb{C}$ and S is a strictly singular operator. In [2], the authors showed that the ideal of strictly singular operators on X_{GM} contains strictly the class of compact operators by constructing a strictly singular operator which is not compact on X_{GM}. In (2007), A. Dehici [8] proved that this operator is quasinilpotent (its spectrum is reduced to the set $\{0\}$). Recall that on some Banach spaces, certain products of strictly singular operators are compact [1]. However, in [2], the authors conjectured that on X_{GM}, we can construct a strictly singular operator such that all of its powers are not compact. Experts in geometry of Banach spaces asked the following question: Does there exist a Banach space X such that every bounded linear operator on X can be written under the form $\lambda I+K$, where $\lambda\in\mathbb{C}$ and K is compact operator? It took fifteen years after the works of T. Gowers and B. Maurey to solve this open complicated problem. Indeed, in [4], the authors were able to construct a non reflexive hereditarily indecomposable Banach space X_{AH} having this property, giving in particular a positive answer to the invariant subspace problem, since on this space, every bounded linear operator which is not scalar-identity is compact or commutes with a non-zero compact operator [18]. The idea of the construction of the space X_{AH} is a combination of HI techniques and the Bourgain-Delbaen construction [7]. After this, Argyros-Motakis [5] have constructed a reflexive HI Banach space denoted by X_{AM} having also a solution to the invariant subspace problem and strictly singular operators on this spaces are not necessarily compact but have compact powers, however the idea of its construction is different from that of the space X_{GM}.

Definition 1.1. An operator $S\in\mathcal{L}(X,Y)$ is said to be strictly singular if for every infinite dimensionalnorm-closed subspace M of X, the restriction of S to M is not an isomorphism.

Let $\mathcal{S}(X,Y)$ denote the set of strictly singular operators from X into Y. It is well known that $\mathcal{S}(X,Y)$ is a norm-closed subspace of $\mathcal{L}(X,Y)$ containing $\mathcal{K}(X,Y)$. For other properties of strictly singular properties we refer to [15].
The properties enjoyed by the space X_{AM} indicated in the end of our introduction are given in the following theorem:

Theorem 1.2. *(see [5]) Banach space X_{AM} has a Schauder basis with the following other properties:*

- $(\imath)$**:** *The space X_{AM} is hereditarily indecomposable,*
- $(\imath\imath)$**:** *For every Y infinite dimensional norm-closed subspace of X_{AM} and every $T\in\mathcal{L}(Y,X_{AM}), T=\lambda I_{Y,X_{AM}}+S$ with S strictly singular,*
- $(\imath\imath\imath)$**:** *For every norm-closed infinite dimensional subspace Y of X_{AM}, the ideal $\mathcal{S}(Y)$ of the strictly singular operators is non separable,*

$(\imath v)$**:** *For every norm-closed infinite dimensional subspace Y of X_{AM} and every $T_1, T_2, T_3 \in \mathcal{S}(Y)$ the operator $T_1T_2T_3$ is compact,*

(v)**:** *For every norm-closed infinite dimensional subspace Y of X_{AM} and every $T \in \mathcal{L}(Y)$, T admits a non-trivial norm-closed invariant subspace.*

Remark 1.3. By taking $Y = X$ in the assertion $(\imath\imath)$, we obtain that for every $A \in \mathcal{L}(X_{AM})$, there exists λ such that $A = \lambda I + S$ where S is strictly singular on X_{AM}. This shows that every bounded linear operator on X_{AM}, or it's a Fredholm operator (with index 0) or is a strictly singular operator. Notice that this property is satisfied also for the case of bounded linear operators defined on X_{GM} as was indicated in the introduction.

Definition 1.4. Let E be a closed (with respect to the operator norm topology) subspace of $\mathcal{L}(X,Y)$. E is said to be have the strong convex compactness property if the following holds: for any finite measure space (Ω,Σ,μ) and any bounded function $U : \Omega \longrightarrow E$ which is strongly measurable (i.e., U(.)x is measurable for all $x \in X$), the strong integral $\displaystyle\int_\Omega U(\omega)d\mu$ (which is an element of $\mathcal{L}(X,Y)$) defined by

$$\int_\Omega U d\mu x = \int_\Omega U(\omega)x d\mu(\omega), x \in X$$

belongs to E.

Example 1.5. The following norm-closed subspaces have the strong convex compactness property

(1) The space $\mathcal{K}(X,Y)$ of compact linear operators [25, 26].
(2) The space $\mathcal{V}(X,Y)$ of completely continuous operators [25].
(3) The space $\mathcal{W}(X,Y)$ of weakly compact linear operators [24].
(4) The space $\mathcal{U}(X,Y)$ of unconditionally summing operators, $\{T \in \mathcal{L}(X,Y)$ for all sequences $(x_n) \subset X$ such that $\displaystyle\sum_n |< x_n, x^\star >| < \infty$ for all $x^\star \in X^\star$, the series $\displaystyle\sum_n T(x_n)$is convergent$\}$ [25].
(5) The space of Dieudonné operators, $\{T \in \mathcal{L}(X,Y)$; for all weak Cauchy sequences $(x_n) \subset X,$ the sequence $T(x_n)$ is weakly convergent$\}$ [25].
(6) The space $\mathcal{S}(X)$ of strictly singular linear operators, if $X = Y = L_p(\mu)$, $1 \leq p \leq \infty$ [26],

Let X,Y be two complex Banach spaces and let $A \in \mathcal{L}(X,Y)$, we define a measure of noncompactness of A by

$$\|A\|_m = \inf\{\|A_{|M}\|, M \text{ norm-closed subspace of } X \text{ with finite codimension}\}$$

It is shown [16] that $\|.\|_m$ is a seminorm on $\mathcal{L}(X,Y)$ with the following property

$$\|A\|_m = 0 \text{ if and only if } A \text{ is compact.}$$

Remark 1.6. Let X,Y be two Banach spaces and let π the canonical homomorphism of $\mathcal{L}(X,Y)$ into $\mathcal{L}(X,Y)/\mathcal{K}(X,Y)$. Thus, for all $A \in \mathcal{L}(X,Y)$, we have

$\|A\|_m \leq \|\pi(A)\|$ but that in general these two seminorms are not equivalent (see, for example, Proposition 2.1 in [17]).

In the following theorem, we mention some estimates satisfied by the measure of noncompactness of strong integrals.

Theorem 1.7. *(see* [10, 26]*) Let X, Y be two Banach spaces, (Ω, μ) be a finite measure space and*

$$U : \Omega \longrightarrow \mathcal{L}(X,Y) \text{ be bounded and strongly measurable}$$

$(\imath)$ **:** *If $X^\star$ is separable, then*

$$\omega \in \Omega \longrightarrow \|U(\omega)\|_m \text{ is measurable}$$

and

$$\|\int_\Omega U(\omega)d\mu(\omega)\|_m \leq \int_\Omega \|U(\omega)\|_m d\mu(\omega) \tag{1.1}$$

$(\imath\imath)$**:** *If X is separable, then*

$$\|\int_\Omega U(\omega)d\mu(\omega)\|_m \leq 2\overline{\int_\Omega}\|U(\omega)\|_m d\mu(\omega) \tag{1.2}$$

$(\imath\imath\imath)$**:** *In general, we have*

$$\|\int_\Omega U(\omega)d\mu(\omega)\|_m \leq 4\overline{\int_\Omega}\|U(\omega)\|_m d\mu(\omega) \tag{1.3}$$

where $\overline{\int_\Omega}$ is the upper integral.

Notice that the strong convex compactness property of $\mathcal{K}(X,Y)$ is an immediate consequence of the inequality (1.1) by taking $\|U(\omega)\|_m$ equals to 0.
In this work, we study the strong convex compactness property on the Banach spaces X_{GM} and X_{AM} for $E =$ the ideal of strictly singular operators on X_{GM} and X_{AM}. This property will arise from a general inequality involving a convenable seminorms defined on these spaces. Moreover, we will give some properties of C_0-semigroups on them and we end this work by some interesting questions.

2. STRONG CONVEX COMPACTNESS PROPERTY FOR THE IDEAL OF STRICTLY SINGULAR OPERATORS ON X_{GM} AND X_{AM}

In this section, we establish our fundamental results in this work by proving the strong convex compactness property for the ideal of strictly singular operators on X_{GM} and X_{AM}. The idea is based on some characterization of these operators by means of a convenable seminorms using the sequences of almost successive vectors and spreading models.

2.1. **The case of the space X_{GM}.** Given two subsets $E_1, E_2 \subset \mathbb{N}$, we say that $E_1 < E_2$ if $\max E_1 < \min E_2$. An interval of integers is a set of the form $\{n, n+1, ..., m\}$. Let $y \in X_{GM}$ and $n \geq 1$, the positive scalar $\|y\|_{(n)}$ is defined by

$$\|y\|_{(n)} = \sup \sum_{i=1}^{n} \|E_i y\|$$

where the sup is extended to all families $E_1 < E_2 ...... < E_n$ of successive intervals (for more details, see [14]). It is known that a seminorm may be defined as follows (see page 68 in [19]):

Let $\mathcal{H}_{X_{GM}}$ be the set of sequences $(x_n)_n$ of almost successive vectors in X_{GM} such that $\limsup \|x_n\|_{(n)} \leq 1$. Let $T \in \mathcal{L}(X_{GM})$, we define $|||T|||_{GM}$ by

$$|||T|||_{GM} = \sup_{x=(x_n)\in\mathcal{H}_{X_{GM}}} \limsup \|T(x_n)\|$$

Proposition 2.1. For every $T \in \mathcal{L}(X_{GM})$, the following assertions are equivalent:

$(\imath)$**:** T is strictly singular,

$(\imath\imath)$**:** $|||T|||_{GM} = 0$.

Proof. For the case $(\imath\imath)$ implies $(\imath)$ (see [19], Lemma 11.2, p. 68). Now, if T is strictly singular, then there exists $\lambda \in \mathbb{C}$ such that $|||T - \lambda I|||_{GM} = 0$ (see [19], p. 81 (the construction of H.I space)), which gives by the first implication that $T - \lambda I$ is strictly singular, but if $\lambda \neq 0$, it follows that T is a Fredholm operator, which is a contradiction, hence we get necessarily that $\lambda = 0$ and consequently, we obtain $|||T|||_{GM} = 0$ which gives the second implication and achieves the proof. $\square$

Theorem 2.2. *Let (Ω, μ) be a finite measure space and $w \in \Omega \longrightarrow U(\omega) \in \mathcal{L}(X_{GM})$ be a strongly integrable function, i.e.,*

$$\widetilde{U}x = \int_\Omega U(\omega) x d\mu(\omega)$$

exists for all $x \in X_{GM}$ as a Bochner integral and

$$\int_\Omega ||U(\omega)|| d\mu(\omega) < \infty$$

If the function $\omega \in \Omega \longrightarrow |||U(\omega)|||_{GM}$ is measurable, then

$$|||\widetilde{U}|||_{GM} \leq \int_\Omega |||U(\omega)|||_{GM} d\mu(\omega) \tag{2.1}$$

Proof. We have by definition

$$|||\widetilde{U}|||_{GM} = \sup_{x=(x_n)\in\mathcal{H}_{X_{GM}}} \limsup \| \int_\Omega U(\omega) x_n d\mu(\omega) \|$$

A classical inequality satisfied by the norm gives that

$$\sup_{x=(x_n)\in\mathcal{H}_{X_{GM}}} \limsup \| \int_\Omega U(\omega) x_n d\mu(\omega) \| \leq \sup_{x=(x_n)\in\mathcal{H}_{X_{GM}}} \limsup \int_\Omega \|U(\omega) x_n\| d\mu(\omega)$$

On the other hand, the functions $w \in \Omega \longrightarrow \|U(\omega)(x_n)\|$ are measurable. The fact that

$$\int_\Omega ||U(\omega)|| d\mu(\omega) < \infty$$

leads (by using the Lebesgue's convergence theorem) to obtain that

$$\limsup \int_\Omega \|U(\omega)x_n\| d\mu(\omega) \leq \int_\Omega \limsup \|U(\omega)x_n\| d\mu(\omega)$$

Hence

$$\begin{aligned}\limsup \int_\Omega \|U(\omega)x_n\| d\mu(\omega) &\leq \int_\Omega \sup_{x=(x_n)\in\mathcal{H}_{X_{GM}}} \limsup \|U(\omega)x_n\| d\mu(\omega)\\ &= \int_\Omega |||U(\omega)|||_{GM} d\mu(\omega)\end{aligned}$$

This gives that

$$\begin{aligned}|||\widetilde{U}|||_{GM} &\leq \int_\Omega \sup_{x=(x_n)\in\mathcal{H}_{X_{GM}}} \limsup \|U(\omega)x_n\| d\mu(\omega)\\ &= \int_\Omega |||U(\omega)|||_{GM} d\mu(\omega)\end{aligned}$$

and achieves the proof. □

As an immediate consequence of Theorem 2.2, we have

Corollary 2.3. Let (Ω,μ) be a finite measure space and $w \in \Omega \longrightarrow U(\omega) \in \mathcal{L}(X_{GM})$ be a strongly integrable function, i.e.,

$$\widetilde{U}x = \int_\Omega U(\omega)x d\mu(\omega)$$

exists for all $x \in X_{GM}$ as a Bochner integral and

$$\int_\Omega ||U(\omega)|| d\mu(\omega) < \infty$$

If the function $\omega \in \Omega \longrightarrow |||U(\omega)|||_{GM}$ is measurable, then
if $U(\omega)$ is strictly singular for almost every $\omega \in \Omega$ on X_{GM}, then $\widetilde{U}$ is strictly singular.

2.2. **The case of the space X_{AM}.** Let X be a Banach space and let $(y_i)_i$ be a seminormalized basic sequence in X and (ε_n) a decreasing sequence of positive real numbers. It is known (see [6]) that there exists a subsequence (x_i) of (y_i) and a seminormalized basic sequence $(\widetilde{x_i})_i$ (in other Banach space) such that for all $n \in \mathbb{N}, (a_i)_{i=1}^n \in [-1,1]^n$ and $n \leq k_1 < ..... < k_n$, one has

$$|||\sum_{i=1}^n a_i x_{k_i}\| - \|\sum_{i=1}^n a_i \widetilde{x_i}||| < \varepsilon_n.$$

The sequence $(\widetilde{x_i})$ is called the spreading model of (x_i) and it is said that (x_i) generates $(\widetilde{x_i})$ as a spreading model. Spreading models of weakly null seminormalized basic sequences have been studied by [3].

Lemma 2.4. *(see* [5]*) Every seminormalized weakly null sequence $\{x_n\}_n$ has a subsequence generating either l_1 or c_0 as a spreading model. Moreover, every infinite dimensional subspace Y of X_{AM} admits both l_1 and c_0 as spreading models.*

The following proposition gives a characterization of strictly singular operators on the space of Argyros-Motakis X_{AM}.

Proposition 2.5. *(see* [5], *Proposition 5.7) Let Y be an infinite dimensional norm-closed subspace of X_{AM} and $T : Y \longrightarrow X_{AM}$ be a linear bounded operator. Then the following assertions are equivalent:*

$(\imath)$**:** *T is not strictly singular,*

$(\imath\imath)$**:** *There exists a basic sequence $(x_k)_{k\in\mathbb{N}}$ in Y generating a c_0 spreading model such that $(T(x_k))_{k\in\mathbb{N}}$ is not norm convergent to 0.*

Let $\mathcal{M}_{X_{AM}}$ be the set of basic sequences $(x_n)_{n=1}^{\infty}$ generating c_0 models in X_{AM}. Now, given $T \in \mathcal{L}(X_{AM})$ and let

$$|||T|||_{AM} = \sup_{x=(x_n)\in\mathcal{M}_{X_{AM}}} \limsup \|T(x_n)\|$$

Our first result in this subsection is the following:

Proposition 2.6. $|||T|||_{AM}$ is a seminorm on X_{AM} satisfying that $|||T|||_{AM} = 0$ if and only if T is strictly singular.

Proof. Let T be strictly singular and assume that $|||T|||_{AM} \neq 0$. By the definition given above, there exists a basic sequence $(y_n)_n$ generating a c_0 spreading model in X_{AM} such that $\limsup \|T(y_n)\| \doteq l \neq 0$, thus $(y_n)_n$ has a subsequence $(y_{n_k})_k$ generating a c_0 spreading model (see [5]) for which $\|T(y_{n_k})_k)\| \nrightarrow 0$ which contradicts the fact that T is strictly singular (see Proposition 2.5). Conversely, if $|||T|||_{AM} = 0$, then $\limsup \|T(y_n)\| \doteq 0$ for all basic sequence $(y_n)_{n=1}^{\infty}$ generating a c_0 spreading model in X_{AM} which implies that $\|T(y_n)\| \longrightarrow 0$. Again by using Proposition 2.5, we get that T is strictly singular which achieves the proof. □

By the same techniques given in the proofs of the case of the Banach space X_{GM}, we can obtain the following results:

Theorem 2.7. *Let (Ω, μ) be a finite measure space and $w \in \Omega \longrightarrow U(\omega) \in \mathcal{L}(X_{AM})$ be a strongly integrable function, i.e.,*

$$\widetilde{U}x = \int_{\Omega} U(\omega)x d\mu(\omega)$$

exists for all $x \in X_{AM}$ as a Bochner integral and

$$\int_{\Omega} ||U(\omega)|| d\mu(\omega) < \infty$$

If the function $\omega \in \Omega \longrightarrow |||U(\omega)|||_{AM}$ is measurable, then

$$|||\widetilde{U}|||_{AM} \leq \int_{\Omega} |||U(\omega)|||_{AM} d\mu(\omega) \tag{2.2}$$

Corollary 2.8. *Let (Ω, μ) be a finite measure space and $w \in \Omega \longrightarrow U(\omega) \in \mathcal{L}(X_{AM})$ be a strongly integrable function, i.e.,*

$$\widetilde{U}x = \int_{\Omega} U(\omega)x d\mu(\omega)$$

exists for exists for all $x \in X_{AM}$ as a Bochner integral and

$$\int_{\Omega} ||U(\omega)||d\mu(\omega) < \infty$$

If the function $\omega \in \Omega \longrightarrow |||U(\omega)|||_{AM}$ *is measurable, then if* $U(\omega)$ *is strictly singular for almost every* $\omega \in \Omega$ *on* X_{AM}, *then* $\widetilde{U}$ *is strictly singular.*

3. C_0-SEMIGROUPS ON X_{GM} AND X_{AM}

In this section, we give a characterization together with some properties of C_0-semigroups defined on X_{AM} and X_{GM}. First of all, we start by giving some basic notions concerning C_0-semigroups which will be used in the following.

Definition 3.1. Let X be a Banach space. A family $(U(t))_{t\geq 0}$ of bounded linear operators from X into X is a semigroup of bounded linear operators on X if

$(\imath)$**:** $U(0) = I$, (I is the identity operators on X),
$(\imath\imath)$**:** $U(t+s) = U(t)U(s)$ for all $t, s \geq 0$,

Moreover, if

$$\lim_{t\longrightarrow 0} U(t)x = x \text{ for every } x \in X$$

$(U(t))_{t\geq 0}$ is called a C_0-semigroup.

Definition 3.2. Let X be a Banach space and let $(U(t))_{t\geq 0}$ be a C_0-semigroup on X with infinitesimal generator T. We define the type ω of $(U(t))_{t\geq 0}$ by

$$\omega = \lim_{t\longrightarrow\infty} \log\frac{||U(t)||}{t}.$$

The spectral bound of T is given by

$$s(T) = \sup_{\lambda\in\sigma(T)} Re\lambda \text{ (where } \sigma(T) \text{ denotes the spectrum of } T).$$

It is known that $s(T) \leq \omega$ and the resolvent of T denoted by $(\lambda - T)^{-1}$ is given as a Laplace transform of $(U(t))_{t\geq 0}$ by the formula

$$(\lambda - T)^{-1} = \int_0^{\infty} e^{-\lambda t}U(t)dt; \qquad Re\lambda > \omega.$$

For more details on the theory of C_0-semigroups on Banach spaces, we can quote for example [21].

Theorem 3.3. *Let* $(U(t))_{t\geq 0}$ *be a* C_0*-semigroup on* X_{AM} *with infinitesimal generator* T. *Thus we have the following two cases:*

$(\imath)$**:** $T \in \mathcal{L}(X_{AM})$,
$(\imath\imath)$**:** $(\lambda - T)^{-1}$ *is compact for all* $\lambda > s(T)$.

Proof. Let $t_0 > 0$. We have two possible cases:

First case: $U(t_0)$ is a Fredholm operator, thus $(U(t))_{t\geq 0}$ can be embedded into a C_0-group of $\mathcal{L}(X_{AM})$ (see Lemma 1 in [12]). Now, the boundedness of T follows from Theorem 2.3 in [22].
Second case: $U(t_0)$ is strictly singular, then by Theorem 1.2 $(\imath v)$, $U(t_0) = [U(\frac{t_0}{3})]^3$ is compact (here $U(\frac{t_0}{3})$ can not be a Fredholm operator since the composition of

Fredholm operators is Fredholm). Thus, for all $t_0 > 0$, $U(t_0)$ is compact on X_{AM}. On the other hand, the fact that $\mathcal{K}(X_{AM})$ has the strict convex compactness property shows that $\int_0^N e^{-\lambda t}U(t)dt$ is compact for all $N \geq 1$ and $\lambda > \omega$. Hence taking the limit in $\mathcal{L}(X_{AM})$, we get

$$\int_0^N e^{-\lambda t}U(t)dt \longrightarrow \int_0^\infty e^{-\lambda t}U(t)dt = (\lambda - T)^{-1}; \qquad \lambda > \omega.$$

which implies that $(\lambda - T)^{-1}$ is compact for $\lambda > \omega$. The compactness for $\lambda > s(T)$ follows from analyticity argument which achieves the proof. □

By using the reflexivity of the spaces X_{GM} and X_{AM} together with Proposition 4.1 in [22], we can obtain the following result concerning the generators of uniformly bounded C_0-semigroups.

Corollary 3.4. *Let T be the generator of a uniformly bounded C_0-semigroup $(U(t))_{t\geq 0}$ on X_{GM} or X_{AM}. Then the following statements are equivalent.*

$(\imath)$ **:** *$KerT$ is infinite dimensional,*

$(\imath\imath)$**:** *T is bounded and a finite rank operator.*

Proposition 3.5. *Let T be the generator of a uniformly bounded C_0-semigroup $(U(t))_{t\geq 0}$ on X_{GM} or X_{AM}. Then $\sigma(T) \bigcap i\mathbb{R}$ is a finite set.*

Proof. Assume that $\sigma(T) \bigcap i\mathbb{R}$ is infinite set. The first part in the proof of Proposition 4.2 in [22] gives that T is bounded. On the other hand, the spectral mapping theorem implies that for t close enough to 1, the bounded linear operator $U(t)$ has an infinite eigenvalues on the unit circle $\{z \in \mathbb{C}/|z| = 1\}$. Now by using Proposition 6.3 in [9], we get that for every $\varepsilon > 0$, the space X_{GM} (resp. X_{AM}) has a $(1+\varepsilon)$-unconditional basic sequence which is a contradiction. Hence the set $\sigma(T) \bigcap i\mathbb{R}$ must be finite. □

Remark 3.6. Notice that the two papers of F. Räbiger and W.J. Ricker [22, 23] are a pioneer works on the subject concerning C_0-semigroups on hereditarily indecomposable Banach spaces, many of their results were improved and refined by [9].

4. Some interesting questions

In finalizing this study, the first question that may come to our mind is the following:

Question 1: Is it true that $\mathcal{S}(X,Y)$ has the strong convex compactness property for every two Banach spaces X and Y? otherwise, what are the structures of Banach spaces such that this property holds?

Definition 4.1. Let X,Y be two Banach spaces and let $F \in \mathcal{L}(X,Y)$. F is called a Fredholm perturbation if $U+F$ is a Fredholm operator whenever U is a Fredholm operator.

The set of Fredholm perturbations is denoted by $\mathcal{F}(X,Y)$ and it was introduced by [11]. In particular, it was shown that $\mathcal{F}(X,Y)$ is a closed subspace of $\mathcal{L}(X,Y)$ containing the subspace of strictly singular operators $\mathcal{S}(X,Y)$.
The same question can be asked for the case of Fredholm perturbations $\mathcal{F}(X,Y)$. More precisely:
Question 2: Is it true that $\mathcal{F}(X,Y)$ has the strong convex compactness property for every two Banach spaces X and Y? otherwise, what are the structures of Banach spaces such that this property holds?

Acknowledgments: I would like to thank the anonymous referees for their pertinent remarks and suggestions. I am also indebted to professors P. Motakis (Texas A&M University, USA) and V. Ferenczi (University of Sao Paulo, Brazil) for the interesting discussions and helpful comments.

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(Received: May 26, 2017)
(Revised: July 20, 2018)

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